AN l^p-VERSION OF VON NEUMANN DIMENSION FOR BANACH SPACE REPRESENTATIONS OF SOFIC GROUPS II

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1. Introduction

This is intended as a follow up paper to [9]. Let us first recall some necessary notation and definitions.

Definition 1.1. Let V be a Banach space. We shall use $\mathrm{Isom}(V)$, for the group of all linear, surjective, isometric maps from V to itself.

Definition 1.2. For $\sigma,\tau\in S_n$, (here S_n is the group of self-bijections of $\{1,\cdots,n\}$) we define the *Hamming distance* by

$$d_{\mathrm{Hamm}}(\sigma,\tau)=\frac{1}{n}|\{j:\sigma(j)\neq\tau(j)\}|,$$

this can also be seen as the probability that $\sigma\neq\tau$, using the uniform probability measure on $\{1,\cdots,n\}$.

Definition 1.3. Let Γ be a countable discrete group. A *sofic approximation* of Γ is a sequence $\sigma_i\colon\Gamma\to S_{d_i}$ of functions, not assumed to be homomorphisms, with $\sigma_i(e)=1$ such that

$$d_{\mathrm{Hamm}}(\sigma_i(gh),\sigma_i(g)\sigma_i(h))\to 0,\ \text{for all } g,h\in\Gamma$$
$$d_{\mathrm{Hamm}}(\sigma_i(g),\sigma_i(h))\to 1,\ \text{for all } g\neq h \text{ in } \Gamma$$

We say that Γ is *sofic*, if it has a sofic approximation.

Definition 1.4. On $M_n(\mathbb{C})$ we shall use $\mathrm{tr}=\frac{1}{n}\mathrm{Tr}$, where Tr is the usual trace. We shall use $\langle A,B\rangle=\mathrm{tr}(B^*A)$, and $\|A\|_p=\mathrm{tr}((A^*A)^{p/2})^{1/p}$. Finally, we shall use $\|A\|_\infty$ for the operator norm of A.

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In fact, if M is a von Neumann algebra with a faithful normal tracial state τ, we will use

$$\|x\|_p = \tau((x^*x)^{p/2})^{1/p}$$

and $\|x\|_\infty$ for the operator norm of $x \in M$.

Definition 1.5. Let Γ be a countable discrete group. An *embedding sequence* of Γ is a sequence $\sigma_i \colon \Gamma \to U(d_i)$, (here $U(n)$ is the unitary group of $\mathbb{C}^n$) so that $\sigma_i(e) = \mathrm{Id}$ and

$$\|\sigma_i(gh) - \sigma_i(g)\sigma_i(h)\|_2 \to 0 \text{ for all } g, h \in \Gamma$$

$$\langle \sigma_i(g), \sigma_i(h)\rangle \to 0 \text{ for all } g \neq h \in \Gamma.$$

We say that Γ is $\mathcal{R}^\omega$*-embeddable* if it has an embedding sequence.

It is a simple exercise to show that every sofic group is $\mathcal{R}^\omega$-embeddable. The class of sofic groups include amenable groups, residually finite groups, and is closed under free products with amalagamation over amenable groups, increasing unions, and taking subgroups. In particular, all linear groups are sofic. If the reader is knowledgeable about operator algebras, then it will not be difficult to show that Γ is $\mathcal{R}^\omega$-embeddable if the group von Neumann algebra $L(\Gamma)$ embeds into an (hence any) ultrapower of the hyperfinite II_1 factor.

In [9] if we are given a countable discrete group Γ and $\Sigma = (\sigma_i \colon \Gamma \to \mathrm{Isom}(V_i))$ with $\dim(V_i) < \infty$, then to every uniformly bounded action of Γ on a Banach space V, we have a number

$$\dim_\Sigma(V, \Gamma) \in [0, \infty].$$

when Γ is a sofic group $V_i = l^p(d_i)$ and σ_i comes from a sofic approximation (also denoted $\sigma_i \colon \Gamma \to S_{d_i},$) and the permuation action of S_n on $l^p(n)$ we denote this number by

$$\dim_{\Sigma, l^p}(V, \Gamma).$$

Similarly, if Γ is $\mathcal{R}^\omega$-embeddable and Σ comes from a embedding sequence (also denoted $\sigma_i \colon \Gamma \to U(d_i)$) we let

$$\dim_{\Sigma, S^p, multi}(V, \Gamma),$$

$$\dim_{\Sigma, S^p, conj}(V, \Gamma)$$

be the dimensions coming from the left multiplication and conjugation actions of $U(n)$ on $S^p(n)$, respectively.

In [9], we proved the following properties of this dimension function

Property 1: $\dim_\Sigma(Y, \Gamma) \leq \dim_\Sigma(X, \Gamma)$ if there is a equivariant bounded linear map $X \to Y$ with dense image.

Property 2: $\dim_\Sigma(V, \Gamma) \leq \dim_\Sigma(W, \Gamma) + \dim_\Sigma(V/W, \Gamma)$, if $W \subseteq V$ is a closed Γ-invariant subspace.

Property 3: $\dim_{\Sigma, l^2}(H, \Gamma) = \dim_{L(\Gamma)} H$ if $H \subseteq l^2(\Gamma \times \mathbb{N})$ is a closed Γ-invariant subspace.

Property 4: $\dim_{\Sigma, l^p}(Y \oplus W, \Gamma) \geq \dim_{\Sigma, l^p}(Y, \Gamma) + \underline{\dim}_{\Sigma, l^p}(W, \Gamma)$ for $2 \leq p < \infty$, where $\underline{\dim}$ is a "lower dimension," and is also an invariant, further

Property 5: $\dim_{\Sigma, l^p}(l^p(\Gamma, V), \Gamma) = \underline{\dim}_{\Sigma, l^p}(l^p(\Gamma, V)) = \dim(V)$ for $1 \leq p \leq 2$.

Thus $\dim_{\Sigma,l^p}$ can be seen as an extension of the von Neumann dimension of a Γ invariant subspace of $l^2(\mathbb{N},\Gamma)$ due to Murray and von Neumann. The above shows that $\dim_{\Sigma,l^p}$ has many of the properties that the usual dimension in linear algebra and the von Neumann dimension have, and this it makes sense to think of $\dim_{\Sigma,l^p}$ as a version of von Neumann dimension. Further in [12] it is shown that we cannot expect *any* invariant for Γ invariant subspaces of $l^p(\Gamma)^{\oplus n}$ to have all the properties of von Neumann dimension. In particular, they show that if Γ conatins an infinite elementary amenable subgroup and $2 < p < \infty$, then there exits closed Γ-invariants subspaces E_n and $F \neq \{0\}$ of $l^p(\Gamma)$ so that $E_n \cap F = \{0\}$ for all n, but

$$l^p(\Gamma) = \bigcup_{n=1}^{\infty} E_n.$$

This is impossible for $p = 2$, because of von Neumann dimension. Taking polars, if $1 < p < 2$, then we can find a closed Γ-invariant subspace $W \subseteq l^p(\Gamma)$, with $W \neq l^p(\Gamma)$ and decreasing closed Γ-invariant subspaces $V_n \subseteq l^p(\Gamma)$ such that $\overline{W + V_n}^{\|\cdot\|_p} = l^p(\Gamma)$ and $\bigcap_{n=1}^{\infty} V_n = \{0\}$. We thus have a Γ-equivariant map with dense image

$$V_n \oplus W \to l^p(\Gamma),$$

so

$$1 \leq \dim_{\Sigma,l^p}(V_n,\Gamma) + \underline{\dim}_{\Sigma,l^p}(W,\Gamma).$$

Thus one of two things occurs. Either

$$\liminf_{n\to\infty} \dim_{\Sigma,l^p}(V_n,\Gamma) > 0$$

or

$$\underline{\dim}_{\Sigma,l^p}(W,\Gamma) \geq 1$$

and W is a proper Γ-invariant subspace of $l^p(\Gamma)$.

Let us briefly recall our definition of dimension.

Definition 1.6. Let Γ be a countable discrete group and $\Sigma = (\sigma_i \colon \Gamma \to \mathrm{Isom}(V_i))$. Let Γ have a uniformly bounded action on a Banach space X, and let $S = (x_j)_{j=1}^{\infty}$ be a bounded sequence in X. For $F \subseteq \Gamma$ finite, and $m \in \mathbb{N}$, let $X_{F,m} = \mathrm{Span}\{gx_j : g \in F^k, 1 \leq j,k \leq m\}$. For $M,\delta > 0$, We let $\mathrm{Hom}_{\Gamma}(S,F,m,\delta,\sigma_i)_M$ consist of all bounded linear maps $T \colon X_{F,m} \to V_i$ such that $\|T\| \leq M$, and

$$\|T(g_1 \cdots g_k x_j) - \sigma_i(g_1)\cdots\sigma_i(g_k)T(x_j)\| < \delta \textit{ for all } g_1,\cdots,g_k \in F, 1 \leq j,k \leq m.$$

We shall typical denote $\mathrm{Hom}_{\Gamma}(S,F,m,\delta,\sigma_i)_1$ by $\mathrm{Hom}_{\Gamma}(S,F,m,\delta,\sigma_i)$. If Σ is a sofic approximation, we use $\mathrm{Hom}_{\Gamma,p}(S,F,m,\delta,\sigma_i)$ for the space of maps above using the permuation action of S_n on $l^p(d_i)$.

Definition 1.7. Let V be a vector space with pseudonorm ρ. For $A \subseteq V$ and $\varepsilon > 0$ we say that a linear subspace $W \subseteq V$ *ε-contains* A, written $A \subseteq_{\varepsilon} W$, if for all $x \in A$, there is a $w \in W$ so that $\rho(x - w) < \varepsilon$. We set $d_{\varepsilon}(A,\rho)$ to be the smallest dimension of a linear subspace which ε-contains A.

Definition 1.8. A *product norm* on $l^{\infty}(\mathbb{N})$ is a norm ρ so that $\rho(f) \leq \rho(g)$ if $|f| \leq |g|$, and so that ρ induces the topology of pointwise convergence on $\{f \in l^{\infty}(\mathbb{N}) : \|f\|_{\infty} \leq 1\}$. If ρ is a product norm, and V is a Banach space we define ρ_V on $l^{\infty}(\mathbb{N},V)$ by $\rho_V(f) = \rho(\|f\|)$.

Definition 1.9. Let Γ, S, Σ, X be as definition 1.6. For $F \subseteq \Gamma$ finite, and $m \in \mathbb{N}$ let $\alpha_S \colon B(X_{F,m}, V_i) \to l^\infty(\mathbb{N}, V_i)$ be given by $\alpha_S(T)(n) = \chi_{j \leq m}(m)T(x_j)$. We define

$$\dim_\Sigma(S, F, m, \delta, \varepsilon, \rho) = \limsup_{i \to \infty} \frac{1}{\dim(V_i)} d_\varepsilon(\alpha_S(\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho_{V_i}).$$

$$\dim_\Sigma(S, \varepsilon, \rho) = \limsup_{(F,m,\delta)} d_\varepsilon(\alpha_S(\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho_{V_i}).$$

$$\dim_\Sigma(S, \rho) = \sup_{\varepsilon > 0} \dim_\Sigma(S, \varepsilon, \rho).$$

Here the triples (F, m, δ) are ordered by $(F, m, \delta) \leq (F', m', \delta')$ if $F \subseteq F', m \leq m', \delta' < \delta$. In [9] it is shown that

$$\dim_\Sigma(S, \rho) = \dim_\Sigma(S', \rho')$$

if $\overline{\mathrm{Span}(\Gamma S)} = \overline{\mathrm{Span}(\Gamma S')}$, and ρ, ρ' are product norms. Because of this we call this number $\dim_\Sigma(X, \Gamma)$ if $\overline{\mathrm{Span}(\Gamma S)} = X$. Also we showed that

$$\dim_\Sigma(X, \Gamma) = \sup_{\varepsilon > 0} \liminf_{(F,m,\delta)} \limsup_{i \to \infty} \frac{1}{\dim(V_i)} d_\varepsilon(\alpha_S(\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho_{V_i}).$$

It is a simple exercise to show that $\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_i)$ may be replaced by $\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_i)_M$ for any $M > 0$.

We let $\underline{\dim}_\Sigma(X, \Gamma)$ be the number obtained by replacing the first limit supremum with a limit infimum (again one can show this depends only on $\overline{\mathrm{Span}(\Gamma S)}$.) Similar to above we showed that

$$\underline{\dim}_\Sigma(X, \Gamma) = \sup_{\varepsilon > 0} \liminf_{(F,m,\delta)} \liminf_{i \to \infty} \frac{1}{\dim(V_i)} d_\varepsilon(\alpha_S(\mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_i), \rho_{V_i}).$$

These definitions may seem quite technical and bizzare, but they are really inspired by ideas of Bowen [3],Kerr and Li [10], Gornay [7] and Voiculescu [19]. The main point is that one should view the usual von Neumann Dimension as a type of dynamical entropy, thinking of the action of Γ on $l^2(\Gamma)$ as an analogue of a Bernoulli shift action of Γ. Bowen in [3], and Kerr and Li in [10] give a microstates version of dynamical entropy for sofic groups. Similar to Kerr and Li, we consider "almost structure-preserving maps" (in this case almost equivariant maps), and measure the growth rate of the size of the space of such maps. Here it makes sense to consider the linear growth rate, since ε-dimension can grow at most linearly.

We prove some of the conjectures stated in [9].We show the following new properties of l^p-Dimension

Property 1: $\underline{\dim}_{\Sigma, l^p}(Y, \Gamma) \geq \dim_{L(\Gamma)}(\overline{Y}^{\|\cdot\|_2})$, when $Y \subseteq l^p(\mathbb{N}, l^p(\Gamma))$ is a Γ-invariant subspace and $1 \leq p \leq 2$.

Property 2: $\underline{\dim}_{\Sigma, S^p}(Y, \Gamma) \geq \dim_{L(\Gamma)}(\overline{Y}^{\|\cdot\|_2})$, when $Y \subseteq l^p(\mathbb{N}, l^p(\Gamma))$ is a Γ-invariant subspace and $1 \leq p \leq 2$.

Property 3: $\dim_{\Sigma, l^p}(X, \Gamma) = 0$ if X is finite-dimensional and Γ is infinite.

Property 4: $\dim_{\Sigma, S^p, multi}\left(\left(\bigoplus_{j=1}^n L^p(L(\Gamma)q_j, \tau)\right), \Gamma\right) = \sum_{j=1}^n \tau(q_j)$, for $1 \leq p < \infty$, where $q_1, \cdots, q_n$ are projections in $L(\Gamma)$ and τ is the group trace.

Our approach to proving the last property is to consider a free, ergodic, probablilty measure-preserving action of Γ, so that the associated equivalence relation $\mathcal{R}_\Gamma$ is sofic (the Bernoulli action for example). Because $\mathcal{R}_\Gamma$ contains an amenable equivalence relation we can try to adapt the proof of the last property in the case $\Gamma = \mathbb{Z}$.

This suggests exploring l^p-Dimension for Representations of Equivalence Relations, which is part of ongoing research.

We also give an equivalent approach to l^p-Dimension defined by using vectors instead of almost equivariant operators.

2. The Issue of Nontriviality

We proceed to show that $\underline{\dim}_{\Sigma,l^p}(Y,\Gamma)$ is nontrivial when $1 \le p \le 2$ and $Y \subseteq l^p(\mathbb{N}, l^p(\Gamma))$ is a Γ-invariant subspace. The idea is as in Section 6 of [9], we extend σ to an embedding sequence $L(\Gamma)$ and look at where the projection onto $\overline{Y}^{\|\cdot\|_2}$ goes. Approximation this projection (thought of as an element in $l^2(\Gamma)$) by elements in Y and applying the appropriate almost equivariant operators will get us the lower bound we want.

Lemma 2.1. *Let H be a Hilbert space, and $\eta_1, \cdots, \eta_k$ an orthonormal system in H, and $V = \operatorname{Span}\{\eta_j : 1 \le j \le k\}$ and P_V the projection onto V. Let K be a Hilbert space and $T \in B(K,H)$ with $\|T\| \le 1$. Then*

$$d_\varepsilon(\{T(\eta_1), \cdots, T(\eta_k)\}) \ge -k\varepsilon + \operatorname{Tr}(P_V T^* T P_V).$$

Proof. For a subspace $E \subseteq H$ we let P_E be the projection onto K. Let W be a subspace of minimal dimension which ε-contains $\{T(\eta_1), \cdots, T(\eta_k)\}$. Then

$$\operatorname{Tr}(P_W TT^*) = \operatorname{Tr}(P_W TT^* P_W) \le \operatorname{Tr}(P_W),$$

similarly

$$\begin{aligned}\operatorname{Tr}(P_W TT^*) &\ge \operatorname{Tr}(P_V T^* P_W T P_V)\\ &= \sum_{j=1}^{k} \langle P_W T(\eta_j), T(\eta_j)\rangle\\ &\ge -\varepsilon k + \sum_{j=1}^{k} \langle T(\eta_j), T(\eta_j)\rangle\\ &= -\varepsilon k + \operatorname{Tr}(P_V T^* T P_V).\end{aligned}$$

□

For convenience, we shall identify $L(\Gamma)$ as a set of vectors in $l^2(\Gamma)$. That is, we shall consider $L(\Gamma)$ to be all $\xi \in l^2(\Gamma)$ so that

$$\|\xi\|_{L(\Gamma)} = \sup_{\substack{f \in c_c(\Gamma),\\ \|f\|_2 \le 1}} \|\xi * f\|_2 < \infty.$$

Here $\xi * f$ is the usual convolution product. By standard arguments, if $\xi \in L(\Gamma)$, then for all $f \in l^2(\Gamma)$, $\xi * f \in l^2(\Gamma)$ and

$$\|\xi * f\|_2 \le \|\xi\|_{L(\Gamma)} \|f\|_2.$$

By general theory, $L(\Gamma)$ is closed under convolution and

$$(\xi * \eta) * \zeta = \xi * (\eta * \zeta)$$

for $\xi, \eta, \zeta \in L(\Gamma)$. Finally for $\xi \in L(\Gamma)$, we set

$$\xi^*(x) = \overline{\xi(x^{-1})}.$$

If $\xi \in L(\Gamma), \zeta, \eta \in l^2(\Gamma)$, then

$$\langle \xi * \eta, \zeta \rangle = \langle \eta, \xi^* * \zeta \rangle.$$

We shall need a few more lemmas, for the first we require the following definitions.

Definition 2.2. We let $\mathbb{C}^*\langle X_1, \cdots, X_n\rangle$ be the free $*$-algebra in n noncommuting variables. That is $\mathbb{C}^*\langle X_1, \cdots, X_n\rangle$ is the universal $\mathbb{C}$-algebra generated by elements $X_1, \cdots, X_n, X_1^*, \cdots, X_n^*$, and we equip $\mathbb{C}^*\langle X_1, \cdots, X_n\rangle$ with a $*$-algebra structure defined on words (and extended by conjugate linearity) by

$$(Y_1 \cdots Y_l)^* = Y_l^* \cdots Y_1^*, Y_j \in \{X_1, \cdots, X_n, X_1^*, \cdots, X_n^*\},$$

here $(X_j^*)^* = X_j$. We call elements of $\mathbb{C}^*\langle X_1, \cdots, X_n\rangle$ $*$-polynomials in n noncommuting variables. Note that if A is a $*$-algebra, and $a_1, \cdots, a_n \in A$, then there is a unique $*$-homomorphism $\mathbb{C}^*\langle X_1, \cdots, X_n\rangle \to A$ sending X_j to a_j. For $P \in \mathbb{C}^*\langle X_1, \cdots, X_n\rangle$, we denote the image under this homomorphism by $P(a_1, \cdots, a_n)$.

Definition 2.3. A *tracial $*$-algebra* is a pair (A, τ) where A is a unital $*$-algebra, $\tau\colon A \to \mathbb{C}$ is a linear map so that $\tau(1) = 1, \tau(x^*x) \geq 0$, with $\tau(x^*x) = 0$ if and only if $x = 0$, and $\tau(xy) = \tau(yx)$ for all $x, y \in A$, and for all $x \in A$, there is a $M > 0$ so that $\tau(y^*x^*xy) \leq M\tau(y^*y)$ for all $y \in A$. An *embedding sequence of* (A, τ) is a sequence of maps $\sigma_i\colon A \to M_{d_i}(\mathbb{C})$ such that

$$\sup_i \|\sigma_i(x)\|_\infty < \infty, \textit{ for all } x \in A$$

$$\sigma_i(1) = 1,$$

$$\operatorname{tr}(\sigma_i(x)) \to \tau(x),$$

$$\|\sigma_i(P(x_1, \cdots, x_n)) - P(\sigma_i(x_1), \cdots, \sigma_i(x_n))\|_2 \to 0$$

for all $x_1, \cdots, x_n \in A$, and $*$ polynomials P in n noncomuting variables. Here $\|x\|_2 = \tau(x^*x)^{1/2}$ for $x \in A$. We let $L^2(A, \tau)$ be the completion of A in $\|\cdot\|_2$. We also let $\pi_\tau\colon A \to B(L^2(A, \tau))$ be given by $\pi_\tau(x)a = xa$, for $x, a \in A$.

The main example which will be relevant for us is $A = c_c(\Gamma)$ with the product being convolution and the $*$-being defined by consider $c_c(\Gamma) \subseteq L(\Gamma)$, and $\tau(f) = f(e)$. Then an embedding sequence of Γ extends to one of $c_c(\Gamma)$ by

$$\sigma_i(f) = \sum_{g \in \Gamma} f(g)\sigma_i(g).$$

Lemma 2.4. *Let (A, τ) be a tracial $*$-algebra. And let M be the weak operator topology closure of $\pi_\tau(A)$ equipped with the trace $\tau(x) = \langle x1, 1\rangle$ for all $x \in M$. Then any embedding sequence of A extends to one of M.*

Proof. By standard arguments, τ is indeed a trace, and since $\{1 \cdot x : x \in A\}$ is dense in A, and elements of M commute with right multiplication it is easy to see that $\tau(x^*x) = 0$ for $x \in M$ if and only if $x = 0$. If $x \in M \setminus A$, by the Kaplansky Density Theorem we may choose a sequence $a_{n,x}$ so that $\|\pi_\tau(a_{n,x})\|_\infty \leq \|x\|_\infty$ and

$$\|a_{n,x} - x\|_2 < 2^{-n}.$$

Note that if ω is a free ultrafilter on $\mathbb{N}$, then σ gives a trace-preserving embedding of A into $\prod^\omega M_{d_i}(\mathbb{C})$, and by uniqueness of GNS representations this extends to a trace-preserving embedding of M into $\prod^\omega M_{d_i}(\mathbb{C})$. So by a standard contradiction and ultrafilter argument, for all $a \in A$, we may find $a_i \in M_{d_i}(\mathbb{C})$ so that $\|a_i\|_\infty \leq \|\pi_\tau(a)\|_\infty$ and $\|a_i - \sigma_i(a)\|_2 \to 0$.

Choose integers $1 \leq i_1 < i_2 < i_3 < \cdots$, and elements $b_{n,x,i} \in M_{d_i}(\mathbb{C})$ so that $\|b_{n,x,i}\|_\infty \leq \|x\|_\infty$ and

$$\|b_{j,x,i} - \sigma_i(a_{j,x})\|_2 < 2^{-n} \text{ for } 1 \leq j \leq n, i \geq i_n.$$

For $x \in M \setminus A$, define $\sigma_i(x) = b_{n,x,i}$ where n is such that $i_n \leq i < i_{n+1}$. If $x \in M \setminus A$, and $i \geq i_n$ we have

$$\|\sigma_i(x) - \sigma_i(a_{n,x})\|_2 \leq 2 \cdots 2^{-n},$$
$$\|\sigma_i(x)\|_\infty \leq \|x\|_\infty.$$

From this estimate it is not hard to see that σ_i is an embedding sequence of M. $\square$

Lemma 2.5. *Let Γ be a countable sofic group, and $\Sigma = (\sigma_i \colon \Gamma \to M_{d_i}(\mathbb{C}))$ a sofic approximation of Γ. Extend σ_i to a embedding sequence, still denoted σ_i, of $(L(\Gamma), \tau)$ with τ the group trace. For $p, q \in \mathbb{N}$ define $\sigma_i \colon M_{p,q}(L(\Gamma)) \to M_{p,q}(M_{d_i}(\mathbb{C}))$ by $\sigma_i(A) = [\sigma_i(a_{lr})]_{1 \leq l \leq p, 1 \leq r \leq q}$. Fix $n \in \mathbb{N}$. For $1 \leq j \leq d_i, 1 \leq k \leq n$ and $E \subseteq \Gamma$ finite define $T_{j,k}^{(E)} \colon l^p(\Gamma)^{\oplus n} \to l^p(d_i)$ by*

$$T_{j,k}^{(E)}(f) = \sum_{g \in E} f_k(g)\sigma_i(g)e_j.$$

Then

(a) For all E and $(1 - o(1))nd_i$ of the j, k we have $\|T_{j,k}^{(E)}\|_{l^p \to l^p} \leq 2$ as $i \to \infty$.

(b) For all $\varepsilon > 0$, for all $f \in c_c(\Gamma), g \in l^p(\Gamma)^{\oplus n}$, there is a finite subset $E \subseteq \Gamma$, so that if $E' \supseteq E$ is a finite subset of Γ, then the set of (j, k) so that

$$\|T_{j,k}^{(E')}(f * g) - \sigma_i(f)T_{j,k}^{(E)}(g)\|_p < \varepsilon,$$

has cardinality at least $(1 - \varepsilon))nd_i$ for all large i.

(c) For all $\varepsilon > 0$, for all $\xi \in M_{1,n}(L(\Gamma))$, (identifying $M_{1,n}(L(\Gamma))$ as a subset of $l^2(\Gamma)^{\oplus n}$ there is a finite subset $E \subseteq \Gamma$, so that if $E' \supseteq E$ is a finite subset of Γ, then the set of (j, k) so that

$$\|T_{j,k}^{(E')}(\xi) - \sigma_i(\xi)(e_j \otimes e_k)\|_2 < \varepsilon,$$

has cardinality at least $(1 - \varepsilon)nd_i$ for all large i.

Proof. (a) This is proved in [9], Theorem 5.3

(b) For $A \in M_{d_i}(\mathbb{C})$,

$$\|A\|_2^2 = \frac{1}{d_i} \sum_{j=1}^{d_i} \|Ae_j\|_2^2,$$

so by Chebyshev's inequality, the fact that $\|T_{j,k}^{(E)}\|_p \leq 1$, and the definiton of embedding sequence, it is enough to verify this for $f = \delta_x, g = \delta_y$ for some $x, y \in \Gamma$. But this is trivial from the definiton of soficity.

(c) Let us first verfiy this when $\xi \in M_{1,n}(c_c(\Gamma))$. In this case, we may again reduce to $\xi = (\delta_{a_1}, \cdots, \delta_{a_k})$ for some $a_1, \cdots, a_k \in \Gamma$. Then if $E \supseteq \{a_1, \cdots, a_k\}$ we have

$$T_{j,k}^{(E)}(\xi) = \sigma_i(a_k)e_j = \sigma_i(\xi)(e_j \otimes e_k).$$

In the general case let $\varepsilon > 0$, given $\xi \in M_{1,n}(L(\Gamma))$ choose $f \in M_{1,n}(c_c(\Gamma))$ so that $\|f - \xi\|_2 < \varepsilon$. Thus for $(1 - (\varepsilon + o(1)))kd_i$ of the (j, k) we have

$$\|T_{j,k}^{(E')}(\xi) - \sigma_i(\xi)(e_j \otimes e_k)\|_2 \leq 2\varepsilon + \|(\sigma_i(\xi) - \sigma_i(f))(e_j \otimes e_k)\|.$$

By the definition of embedding sequence for all large i we have

$$\frac{1}{d_i}\sum_{j=1}^{d_i}\sum_{k=1}^{n}\|(\sigma_i(\xi)-\sigma_i(f))(e_j\otimes e_k)\|_2^2<\varepsilon^2,$$

thus for at least $(1-\sqrt{\varepsilon})nd_i$ of the (j,k) we have

$$\|(\sigma_i(\xi)-\sigma_i(f))(e_j\otimes e_k)\|_2<\sqrt{\varepsilon},$$

combining these estimates completes the proof.

□

Proposition 2.6. *Let Γ be a countable discrete group, let $1\leq p\leq 2$, and Y a closed Γ-invariant subspace of $l^p(\mathbb{N},l^p(\Gamma))$, with Γ acting by $gf(x)=f(g^{-1}x)$. Set $H=\overline{Y}^{\|\cdot\|_2}$.*

(a) Suppose Σ is a sofic approximation of Γ, then

$$\underline{\dim}_{\Sigma,l^p}(Y,\Gamma)\geq\dim_{L(\Gamma)}(H).$$

(b)Suppose Σ is a embedding sequence of Γ, then

$$\underline{\dim}_{\Sigma,S^p}(Y,\Gamma)\geq\dim_{L(\Gamma)}(H).$$

Proof. By general theory, we can find vectors $(\xi^{(q)})_{q=1}^{\infty}\in H$, so that

$$\langle\lambda(g)\xi^{(s)},\xi^{(s)}\rangle=q_s(g^{-1}),\ \text{where } q_s \text{ is a projection in } L(\Gamma),$$

$$\sum_{s=1}^{\infty}\tau(q_s)=\dim_{L(\Gamma)}\mathcal{H},$$

$$\langle\lambda(g)\xi^{(j)},\xi^{(l)}\rangle=0 \text{ for } j\neq l, g\in\Gamma.$$

$$H=\bigoplus_{j=1}^{\infty}\overline{L(\Gamma)\xi^{(j)}}.$$

These equations can be rewritten as

$$\sum_{i=1}^{n}\xi^{(j)}*(\xi^{(j)})^*=q_j,\ \text{for } 1\leq j\leq\infty$$

$$\sum_{i=1}^{n}\xi^{(j)}*(\xi^{(l)})^*=0 \text{ if } j\neq l,$$

Let us illuminate these equations a little. Regard a vector $\xi\in l^2(\Gamma)^{\oplus n}$ as a element in $M_{1,n}(l^2(\Gamma))$ with the product of two matrices induced from convolution of vectors. Then the product of elements of $M_{1,n}(l^2(\Gamma)),M_{n,1}(L(\Gamma))$ makes sense, but may not land back in $l^2(\Gamma)$. The above equations then read

$$\xi^{(j)}(\xi^{(j)})^*=q_j\text{for } 1\leq j<\infty,$$

$$\xi^{(j)}(\xi^{(l)})=0 \text{ for } j\neq l..$$

In particular, the above equations imply that

$$\|\xi_r^{(j)}\|_{L(\Gamma)}\leq 1.$$

So that $\xi^{(j)}\in M_{1,n}(L(\Gamma))$. Extend σ_i to a embedding sequence of $M_{n,m}(L(\Gamma))$ for all n,m and such that

$$\|\sigma_i(\xi^{(j)})\|\leq 1,\ \text{for all } j$$

$$\|\sigma_i(\xi_r^{(j)})\| \leq 1, \text{ for all } j, r$$
$$\sigma_i(\xi^{(j)})\sigma_i(\xi^{(l)})^* = 0 \text{ for all } j \neq l.$$

for all j, r.

(a)

Let $S = (x_j)_{j=1}^n$ be a dynamical generating sequence for Y.

Fix $\eta > 0, t \in \mathbb{N}$ and choose a finite subset $F_1 \subseteq \Gamma, m_1 \in \mathbb{N}$, and $c_{gj}^{(s)}$ for $1 \leq s \leq t, (g, j) \in F_1 \times \{1, \cdots, m_1\}$ so that for all $1 \leq s \leq t$

$$\left\| \xi^{(s)} - \sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(s)} g x_j \right\|_2 < \eta.$$

Choose finitely supported functions x_j' so that $\|x_j - x_j'\|_p < \eta'$. Since $p \leq 2$, it is easy to see that if we force η' to be sufficiently small then,

$$\left\| \xi^{(s)} - \sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(s)} g x_j' \right\|_2 < \eta.$$

Let $S = (x_j)_{j=1}^\infty$ be a dynamically generating sequence for Y. Fix $F \subseteq \Gamma$ finite $m \in \mathbb{N}, \delta > 0$. Let $E \subseteq \Gamma$ be finite, let $T_{j,k}^{(E)}$ be defined as in the preceding lemma.

It is easy to see that if E is sufficently large, then $T_{j,k}^{(E)}\big|_{Y_{F,m}} \in \operatorname{Hom}_\Gamma(S, F, m, \delta, \sigma_i)_2$ for $(1 - o(1))nd_i$ of the j, k, and in fact $\|T_{j,k}^{(E)}\|_{l^p \to l^p} \leq 2$ for $1 \leq p \leq 2$. For such (j, k), and for all small δ, for $1 \leq s \leq t + 1$

$$\left\| T_{j,k}^{(E)}(\xi^{(s)}) - \sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(p)} \sigma_i(g) T_{j,k}^{(E)}(x_j) \right\|_2 < 2\eta,$$
$$\|T_{j,k}^{(E)}(g x_j') - T_{j,k}^{(E)}(g x_j)\|_2 < \eta.$$

Thus by the preceding lemma, for at least $(1 - (2013)!\varepsilon)nd_i$ of the j, k we have

$$\left\| \sigma_i(\xi^{(s)})(e_j \otimes e_k) - \sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(p)} \sigma_i(g) T_{j,k}^{(E)}(x_j) \right\|_2 < \varepsilon + \eta.$$

Now consider the linear map $A\colon l^\infty(\mathbb{N}, l^p(d_i)) \to l^2(d_i)^{\oplus t}$ given by

$$S(f) = \left(\sum_{\substack{g \in F_1 \\ 1 \leq j \leq m_1}} c_{gj}^{(p)} \sigma_i(g) f(j) \right)_{p=1}^t,$$

from the above it is easy to see that if $\alpha_S(\operatorname{Hom}_\Gamma(S, F, m, \delta, \sigma_i)) \subseteq_{\varepsilon'} V$ and ε' is sufficiently small,

$$A(V) \supseteq_{\varepsilon, \|\cdot\|_2} \{\phi_i(e_j \otimes e_k) : (j, k) \in A_i\},$$

with

$$\frac{|A_i|}{d_i} \to (1-(2013)!\varepsilon)nd_i,$$

$$\phi_i(f) = (\sigma_i(\xi^{(1)})(f), \sigma_i(\xi^{(2)})(f), \cdots, \sigma_i(\xi^{(t)})(f)).$$

Thus ϕ_i is given in matrix form by

$$\phi_i = \begin{bmatrix} \sigma_i(\xi^{(1)}) & 0 & \cdots & 0 \\ 0 & \sigma_i(\xi^{(2)}) & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_i(\xi^{(t)}) \end{bmatrix}.$$

As

$$\phi_i\phi_i^* = \begin{bmatrix} \sigma_i(\xi^{(1)})\sigma_i(\xi^{(1)})^* & 0 & \cdots & 0 \\ 0 & \sigma_i(\xi^{(2)})\sigma_i(\xi^{(2)})^* & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma_i(\xi^{(t)})\sigma_i(\xi^{(t)})^* \end{bmatrix}$$

By our choice of σ_i we have

$$\|\phi_i\| \leq 1,$$

By Lemma 2.1, we find that

$$\dim_{\Sigma,l^p}(V,\Gamma) \geq (1-(2013)!\varepsilon)n + \dim_{L(\Gamma)} H_t.$$

Letting $\varepsilon \to 0, t \to \infty$ completes the proof.

(b) Same proof as in (a). $\square$

3. A Complete Calculation In The Case of $\bigoplus_{j=1}^n L^p(L(\Gamma))q_j$.

In this section, we show that if Γ is $\mathcal{R}^\omega$-embeddable, Σ is an embedding sequence and $q_1, \cdots, q_n \in \mathrm{Proj}(L(\Gamma))$, then

$$\dim_{\Sigma,S^p,multi}\left(\bigoplus_{j=1}^n L^p(L(\Gamma),\tau)q_j, \Gamma\right) = \underline{\dim}_{\Sigma,S^p,multi}\left(\bigoplus_{j=1}^n L^p(L(\Gamma),\tau)q_j, \Gamma\right) = \sum_{j=1}^n \tau(q_j)$$

where τ is the group trace. In [9], we only showed this for $2 \leq p < \infty$, we now complete the proof for all p.

We shall frequently use functional calculus throughout the proofs. Finally for notation in this section we use $\{u_g : g \in \Gamma\}$ for the canonical unitaries generating $L(\Gamma)$.

Lemma 3.1. *(a) Let $n \in \mathbb{N}$, suppose that $A, B \in M_n(\mathbb{C})$ are such that $|A| \leq |B|$, then for all $\beta > 0$,*

$$\mathrm{tr}(|A|^\beta) \leq \mathrm{tr}(|B|^\beta).$$

(b) Suppose that $A, B \in M_n(\mathbb{C})$ and Q is a orthogonal projection in $M_n(\mathbb{C})$. Fix $1 \leq p < \infty$, suppose that $\delta, \eta > 0$ are such that

$$\|(A-1)B\|_p < \delta, \|A-Q\|_p < \eta.$$

Then

$$\|B - \chi_{(0,\sqrt{\delta})}(|A-1|)B\|_p < \sqrt{\delta},$$

and

$$\operatorname{tr}(\chi_{(0,\sqrt{\delta})}(|A-1|)) \leq \operatorname{tr}(Q) + \left(\frac{\eta}{1-\sqrt{\delta}}\right)^p.$$

Proof. We first make the following preliminary observation: if P, Q are orthogonal projections in $M_n(\mathbb{C})$ with

$$P\mathbb{C}^n \cap Q\mathbb{C}^n = \{0\},$$

then

$$\operatorname{tr}(P) \leq 1 - \operatorname{tr}(Q).$$

This follows directly from the fact that $1 - Q$ is injective on $P\mathbb{C}^n$.

(a) First note that

$$\operatorname{tr}(T^\alpha) = \alpha \int_0^\infty t^{\alpha-1} \operatorname{tr}(\chi_{(t,\infty)}(T))\, dt$$

if $T \geq 0$. Now note that if $0 \leq T \leq S$, and

$$\xi \in \chi_{(t,\infty)}(T)(\mathbb{C}^n) \cap \chi_{[0,t]}(S)(\mathbb{C}^n)$$

and $\xi \neq 0$, then

$$t\|\xi\|^2 < \langle T\xi, \xi\rangle \leq \langle S\xi, \xi\rangle \leq t\|\xi\|^2,$$

which is a contradiction. Hence

$$\xi \in \chi_{(t,\infty)}(T)(\mathbb{C}^n) \cap \chi_{[0,t]}(S)(\mathbb{C}^n) = \{0\},$$

so the above integral formula and our preliminary observation prove (a).

(b) Note that

$$\begin{aligned} |\chi_{[\sqrt{\delta},\infty)}(|A-1|)B|^2 &= B^*\chi_{[\sqrt{\delta},\infty)}(|A-1|)B \\ &\leq \frac{1}{\delta} B^*|A-1|^2 B = |\frac{1}{\sqrt{\delta}}(A-1)B|^2, \end{aligned}$$

thus by (a)

$$\|B - \chi_{(0,\sqrt{\delta})}(|A-1|)B\|_p = \|\chi_{[\sqrt{\delta},\infty)}(|A-1|)B\|_p < \sqrt{\delta}.$$

Further if

$$\xi \in \chi_{(0,\sqrt{\delta})}(|A-1|)(\mathbb{C}^n) \cap (1-Q)(\mathbb{C}^n) \cap \chi_{[0,1-\sqrt{\delta}]}(|A-Q|)(\mathbb{C}^n),$$

is nonzero, then

$$(1-\sqrt{\delta})^2\|\xi\|^2 \geq \langle |A-Q|^2\xi, \xi\rangle = \|A\xi\|^2 > (1-\sqrt{\delta})^2\|\xi\|^2,$$

which is a contradiction. Thus

$$\operatorname{tr}(\chi_{(0,\sqrt{\delta})}(|A-1|)) \leq \operatorname{tr}(Q) + \operatorname{tr}(\chi_{(1-\sqrt{\delta},\infty)}(|A-Q|)).$$

Since

$$\chi_{(1-\sqrt{\delta},\infty)}(|A-Q|) \leq \frac{|A-Q|^p}{(1-\sqrt{\delta})^p},$$

we have that

$$\operatorname{tr}(\chi_{(1-\sqrt{\delta},\infty)}(|A-Q|)) < \frac{\eta^p}{(1-\sqrt{\delta})^p}.$$

$\square$

Proposition 3.2. *Let Γ be an $\mathcal{R}^\omega$-embeddable group and Σ an embedding sequence. Let $M = L(\Gamma)$ and τ the canonical group trace on M. Then, for all $1 \le p < \infty$ and for every $q \in \mathrm{Proj}(M)$ we have*

$$\underline{\dim}_{\Sigma,S^p,\mathrm{multi}}\left(\bigoplus_{j=1}^n L^p(M,\tau)q_j,\Gamma\right) \le \sum_{j=1}^n \tau(q_j).$$

Proof. By subadditivity of dimension, it suffices to handle the case of $L^p(M,\tau)q$. Let $0 < \varepsilon, \kappa < 1/2$. Let A be the $*$-algebra in $L(\Gamma)$ generated by q and Γ, let

$$\widetilde{\sigma_i}\colon A \to M_{d_i}(\mathbb{C})$$

be an embedding sequence which extends σ_i, and choose projections q_i so that $\|q_i - \sigma_i(q)\|_p \to 0$. Choose $f \in c_c(\Gamma)$ so that

$$\left\|q - \sum_{s\in\Gamma} f(s)u_s\right\|_p < \kappa.$$

If $T\colon L^p(M,\tau)q \to L^p(M_{d_i}(\mathbb{C}),\mathrm{tr})$, define

$$\widetilde{T}(x) = T(xq).$$

Let F be the support of f, then if $m \in \mathbb{N}, \kappa, \delta > 0$ are sufficiently small we have

$$\left\|\left(\sum_{s\in\Gamma} f(s)\sigma_i(s) - 1\right)\widetilde{T}(q)\right\|_p < \varepsilon^2,$$

for all $\widetilde{T} \in \mathrm{Hom}_\Gamma(S,F,m,\delta,\sigma_i)$. Thus the proceeding lemma implies that if

$$e_i = \chi_{(\varepsilon,\infty)}\left(\left|\sum_{s\in\Gamma} f(s)\sigma_i(s) - 1\right|\right),$$

then for all large i, we have

$$\|T(q) - e_iT(q)\|_p < \varepsilon,$$

$$\mathrm{tr}(e_i) \le \mathrm{tr}(q_i) + 2^p\kappa^p \to \tau(q) + 2^p\kappa^p.$$

Since $\kappa > 0$ is arbitrary, this proves the claim.

□

Lemma 3.3. *Fix $1 \le p \le \infty$, and a sequence of positive integers $d(n) \to \infty$, and let μ_n be the Lebesgue measure on $L^p(M_{d(n)}(\mathbb{C}), \frac{1}{d(n)}\mathrm{Tr})$ normalized so that $\mu_n(\mathrm{Ball}(L^p(M_{d(n)}(\mathbb{C}), \frac{1}{d(n)}\mathrm{Tr}))) = 1$. Let $A_n \subseteq \mathrm{Ball}(V_n)$ and suppose that there is an $\alpha > 0$ so that*

$$\limsup_{n\to\infty} \mu_n(\mathrm{Ball}(V_n))^{1/2d(n)^2} \ge \alpha > 0.$$

Further, let $q_n \in \mathrm{Proj}(M_{d(n)}(\mathbb{C}))$ be such that $\frac{1}{d(n)}\mathrm{Tr}(p_n)$ converges to a positive real number. Then for every $\varepsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{d(n)\,\mathrm{Tr}(q_n)} d_\varepsilon(A_nq_n, \|\cdot\|_p) \ge \kappa(\alpha,\varepsilon)$$

with

$$\lim_{\varepsilon\to 0} \kappa(\alpha,\varepsilon) = 1 \text{ for all fixed } \alpha > 0.$$

Proof. Fix $1 > \varepsilon > 0$, and suppose that

$$\limsup_{n\to\infty} \frac{1}{d(n)\operatorname{Tr}(p_n)} d_\varepsilon(A_n q_n, \|\cdot\|_p) < \kappa,$$

then for all large n,

$$d_\varepsilon(A_n q_n, \|\cdot\|_{V_n}) < d(n)\kappa \operatorname{Tr}(q_n).$$

Let W_n be a subspace of dimension at most $d(n)\kappa\operatorname{Tr}(q_n)$ which ε-contains $A_n p_n$, thus

$$A_n q_n \subseteq (1+\varepsilon)\operatorname{Ball}(W_n) + \varepsilon \operatorname{Ball}(L^p(M_{d(n)}(\mathbb{C}), \frac{1}{d(n)}\operatorname{Tr})q_n).$$

Let $S \subseteq (1+\varepsilon)\operatorname{Ball}(W_n)$ be a maximal family of ε-separated vectors, i.e. for all $x, y \in S$ with $x \neq y$ we have $\|x-y\| \geq \varepsilon$. Then the $\varepsilon/3$ balls centered at points in S are disjoint so by a volume computation

$$|S| \leq \left(\frac{3+3\varepsilon}{\varepsilon}\right)^{2\dim(W_n)}.$$

By maximality, S is ε-dense in $(1+\varepsilon)\operatorname{Ball}(W_n)$. Thus

$$A_n q_n \subseteq \bigcup_{x\in S} x + 2\varepsilon \operatorname{Ball}(L^p(M_{d(n)}(\mathbb{C}), \frac{1}{d(n)}\operatorname{Tr})q_n),$$

so

$$\operatorname{vol}(A_n q_n) \leq 2^{2d(n)\operatorname{Tr}(p_n)} \varepsilon^{2d(n)\operatorname{Tr}(q_n) - 2\dim(W_n)\operatorname{Tr}(q_n)} (3+3\varepsilon)^{2\dim(W_n)} V_p(q_n),$$

where for $q \in \operatorname{Proj}(M_{d(n)}(\mathbb{C}))$ we use

$$V_p(q) = \operatorname{Ball}\left(L^p(M_{d(n)}(\mathbb{C}), \frac{1}{d(n)}\operatorname{Tr})q\right).$$

Since $A_n \subseteq A_n q_n \times \operatorname{Ball}\left(L^p(M_{d(n)}(\mathbb{C}), \frac{1}{d(n)}\operatorname{Tr}\right)$, we have

$$\alpha \leq \limsup_{n\to\infty} 6 \cdot 2^{\frac{1}{d(n)}\operatorname{Tr}(q_n)} \varepsilon^{(1-\kappa)\frac{\operatorname{Tr}(q_n)}{d(n)}} \left(\frac{V(q_n)V(1-q_n)}{V(\operatorname{Id}_{d(n)})}\right)^{1/2d(n)^2}$$

Hence it suffices to show that

$$\limsup_{n\to\infty} \left(\frac{V_p(q_n)V_p(1-q_n)}{V_p(\operatorname{Id}_{d(n)})}\right)^{1/2d(n)^2} < \infty.$$

For this, we know that there is a constant $C > 0$ so that

$$\left(\frac{V_p(q_n)V_p(1-q_n)}{V_p(\operatorname{Id}_{d(n)})}\right)^{1/2d(n)^2} \leq C\left(\frac{V_p(q_n)V_p(1-q_n)}{V_2(\operatorname{Id}_{d(n)})}\right)^{1/2d(n)^2}.$$

Further by the Santolo inequality ,we may reduce to the case that $p \geq 2$. Since $\|A\|_2 \leq \|A\|_p$, we then have that

$$\left(\frac{V_p(q_n)V_p(1-q_n)}{V_2(\operatorname{Id}_{d(n)})}\right)^{1/2d(n)^2} \leq \left(\frac{V_2(q_n)V_2(1-q_n)}{V_2(\operatorname{Id}_{d(n)})}\right)^{1/2d(n)^2}.$$

Since

$$V_2(q) = \frac{\pi^{\operatorname{Tr}(q)}}{\operatorname{Tr}(q)!} d(n)^{-d(n)},$$

it follows from Stirling's formula and the fact that $\frac{1}{d(n)}\operatorname{Tr}(q_n)$ converges that

$$\left(\frac{V_2(q_n)V_2(1-q_n)}{V_2(\mathrm{Id}_{d(n)})}\right)^{1/2d(n)^2}$$

is bounded.

$\square$

To complete the calculation, it suffices to prove the following Theorem.

Theorem 3.4. *Let Γ be an $\mathcal{R}^\omega$-embeddable group and Σ an embedding sequence. Let $M = L(\Gamma)$ and τ the canonical group trace on M. Then, for all $1 \leq p < \infty$ and for every $q \in \operatorname{Proj}(M)$ we have*

$$\dim_{\Sigma,S^p,\mathrm{multi}}\left(\bigoplus_{j=1}^n L^p(M,\tau)q_j,\Gamma\right) = \underline{\dim}_{\Sigma,S^p,\mathrm{multi}}\left(\bigoplus_{j=1}^n L^p(M,\tau)q_j,\Gamma\right)$$

$$= \sum_{j=1}^n \tau(q_j).$$

Proof. We use the generating sequence $S = (q_1, \cdots, q_n, 0, \cdots)$ to do the calculation. By Proposition 3.2, we have the upper bound. So it suffices to prove the lower bound. Let R be the $*$-algebra in $L(\Gamma)$ generated by Γ and $\{q_1, \cdots, q_n\}$. By Lemma 6.3 in [9], we can find a sequence of linear, asymptotically trace-preserving, asymptotic $*$-homomorphisms

$$\rho_i \colon R \to M_{d_i}(\mathbb{C})$$

such that

$$\rho_i(u_g) = \sigma_i(g),$$

and

$$\sup_i \|\rho_i(x)\|_\infty < \infty$$

for all $x \in R$.

Choose projections in $f_i^{(j)} \in M_{d_i}(\mathbb{C})$ so that

$$\|f_i^{(j)} - \rho_i(q_j)\|_p \to 0.$$

Fix $F \subseteq \Gamma$ finite $m \geq n$ in $\mathbb{N}$, $\delta > 0$. Let $E \subseteq \Gamma$ be a finite set which is sufficently large in a manner to be determined later. Let

$$V_E^{(j)} = \operatorname{Span}\{u_g q : g \in E\}.$$

For $A \in M_{d_i}(\mathbb{C})$ define

$$T_A^{(j)}\left(\sum_{g\in E} c_g u_g q\right) = \sum_{g\in E} c_g \sigma_i(g) f_i^{(j)} A.$$

As in Proposition 7.3 in [9] it follows that if E is sufficiently large then for all large i,

$$T_{A_1}^{(1)} \oplus \cdots \oplus T_{A_n}^{(n)} \in \operatorname{Hom}_\Gamma(S, F, m\delta, \sigma_i)_n$$

for all $A_1, \cdots, A_n$ with $\|A_j\|_\infty \leq 1$. Since

$$\left(\frac{\mathrm{vol}(\mathrm{Ball}(M_{d_i}(\mathbb{C}), \|\cdot\|_\infty)}{\mathrm{vol}(\mathrm{Ball}(M_{d_i}(\mathbb{C}), \|\cdot\|_{L^p(1/d_i\,\mathrm{Tr})})}\right)^{1/2d_i^2},$$

The theorem follows from the preceding Lemma.

□

If Γ is a countable discrete group we use $C^*_\lambda(\Gamma)$ for $\overline{\mathbb{C}[\Gamma]}^{\|\cdot\|_\infty}$, with the clousre taken in the left regular representation. As a corollary of the above Theorem, we deduce one of conjectures stated in [9].

Corollary 3.5. *Let Γ be an $\mathcal{R}^\omega$-embeddable group and $1 \leq p < \infty$. Let $I \subseteq C^*_\lambda(\Gamma)$ be a norm closed left-ideal. Let $\overline{I}^{\mathrm{wk}^*} = L(\Gamma)q$ (with the closure taken in $L(\Gamma)$. Then*

$$\dim_{\Sigma, S^p, \mathrm{multi}}(I, \Gamma) \geq \tau(q).$$

Proof. It suffices to show that the inclusion $I \subseteq L^p(L(\Gamma), \tau)q$ has dense image. Let $q' \in \mathrm{Proj}(L(\Gamma))$ be such that

$$\overline{I}^{\|\cdot\|_p} = L^p(L(\Gamma), \tau)q'.$$

By the argument in [9], Proposition 7.6,

$$q' = \sup_{x \in I} \chi_{(0,\infty)}(|x|).$$

So it suffices to prove the following two claims.

Claim 1. If $x \in C^*_\lambda(\Gamma)$, then $\chi_{(0,\infty)}(|x|) \in \overline{I}^{\mathrm{wk}^*}$.

Claim 2. If $e, f \in \mathrm{Proj}(\overline{I}^{\mathrm{wk}^*})$, then $e \vee f \in \mathrm{Proj}(\overline{I}^{\mathrm{wk}^*}$.

For the proof of claim I, let $x = v|x|$ be the polar decomposition. By the Kaplansky Density Theorem, we can find $v_n \in C^*_\lambda(\Gamma)$ so that $\|v_n\|_\infty \leq 1$ and $\|v_n - v\|_2 \to 0$. But then $\|v_n^* x - |x|\|_2 \to 0$, so $|x| \in \overline{I}^{\mathrm{wk}^*}$. Since

$$\chi_{(\varepsilon,\infty)}(|x|) = |x|^{-1}\chi_{(\varepsilon,\infty)}(|x|)|x|,$$

we find that $\chi_{(0,\infty)}(|x|) \in \overline{I}^{\mathrm{wk}^*}$.

For the proof of claim 2, we use the formula (proved by functional calculus):

$$e \vee f = 1 - \lim_{n\to\infty}([(1-e)(1-f)(1-e)])^n$$

where the limit is in $\|\cdot\|_2$. Since $e, f \in L(\Gamma)q$, a little calculcation shows that

$$1 - ([(1-e)(1-f)(1-e)])^n \in L(\Gamma)q,$$

this proves the corollary.

□

We can also handle the case $p = \infty$ if we assume a little more.

Theorem 3.6. *Let Γ be a countable discrete group. Assume that there is a sequence $\Sigma = (\sigma_i \colon \mathbb{C}[\Gamma] \to M_{d_i}(\mathbb{C}))$ of linear maps such that for all $*$-polynomials in n noncommuting variables P, and for all $x_1, \cdots, x_n \in \mathbb{C}[\Gamma]$ we have*

$$\|P(\sigma_i(x_1), \cdots, \sigma_i(x_n)) - \sigma_i(P(x_1, \cdots, x_n))\|_\infty \to 0,$$

and

$$\|\sigma_i(x)\|_\infty \to \|x\|_{C^*_\lambda(\Gamma)},$$

$$\operatorname{tr}(\sigma_i(x)) \to \tau(x)$$

for all $x \in \mathbb{C}[\Gamma]$*. Let* $I \subseteq C^*_\lambda(\Gamma)$ *be a norm-closed left ideal, and let* $I^{\mathrm{wk}^*} = L(\Gamma)q$*, with* $q \in \operatorname{Proj}(L(\Gamma))$*. Then,*

$$\underline{\dim}_{\Sigma,S^\infty,\mathrm{multi}}(I,\Gamma) \geq \tau(q).$$

Proof. Let

$$A = \frac{\{(x_i)_{i=1}^\infty \in \prod_{i=1}^\infty M_{d_i}(\mathbb{C}) : \sup_i \|\sigma_i(x)\|_\infty < \infty\}}{\{(x_i)_{i=1}^\infty \in \prod_{i=1}^\infty M_{d_i}(\mathbb{C}) : \sup_i \|\sigma_i(x)\|_\infty \to 0\}},$$

then our hypothesis implies that there is an isometric $*$-homomorphism

$$\sigma\colon \mathbb{C}[\Gamma] \to A,$$

such that

$$\sigma(u_g) = \pi(\sigma_1(g), \sigma_2(g), \cdots)$$

where

$$\pi\colon \left\{(x_i)_{i=1}^\infty \in \prod_{i=1}^\infty M_{d_i}(\mathbb{C}) : \sup_i \|\sigma_i(x)\|_\infty < \infty\right\} \to A,$$

is the quotient map. We may extend σ to an injective $*$-homomoprhism $\sigma\colon C^*_\lambda(\Gamma) \to A$. Let $S = (x_n)_{n=1}^\infty$ be a dynamically generating sequence for I. Let

$$\phi_i\colon C^*_\lambda(\Gamma) \to M_{d_i}(\mathbb{C})$$

be such that

$$\sigma(x) = \pi((\phi_1(x), \phi_2(x), \cdots)).$$

As before, we may extend ϕ_i to an embedding sequence

$$\psi_i\colon L(\Gamma) \to M_{d_i}(\mathbb{C}).$$

Now let $\varepsilon > 0$, and choose a finite subset $E \subseteq \Gamma, l \in \mathbb{N}$, and $c_{gj} \in \mathbb{C}$, for $(g,j) \in E \times \{1, \cdots, l\}$ so that

$$\left\| q - \sum_{\substack{g \in E \\ ,1 \leq j \leq l}} c_{gj} u_g x_j \right\|_2 < \varepsilon.$$

Fix $E \subseteq F \subseteq \Gamma$ finite, $l \leq m \in \mathbb{N}, \delta > 0$. Since all injective $*$-homorphisms defined on C^*-algebras are isometric, it is easy to see that if we define $\rho_i = \frac{\phi_i\big|_{I_{F,m}}}{\left\|\phi_i\big|_{I_{F,m}}\right\|}$, then

$$\left\|\rho_i - \phi_i\big|_{I_{F,m}}\right\| \to 0.$$

For $B \in M_{d_i}(\mathbb{C})$ define

$$T_B\colon I_{F,m} \to M_{d_i}(\mathbb{C}),$$

by

$$T_B(x) = \rho_i(x)B.$$

If $\|B\|_\infty \leq 1$, then

$$\|T_B(x)\| \leq \|B\|_\infty.$$

Further if $\|B\|_\infty \leq 1$, and $1 \leq j,k \leq m$, and $g_1, \cdots, g_k \in F$, then

$$\begin{aligned}&\|T_B(g_1\cdots g_kx_j)-\sigma_i(g_1)\cdots\sigma_i(g_k)T_B(x_j)\|\\&\leq\|\phi_i(g_1\cdots g_kx_j)-\sigma_i(g_1)\cdots\sigma_i(g_k)\phi_i(x_j)\|\\&\to 0\end{aligned}$$

using that

$$\pi((\phi_i(g_1\cdots g_kx_j))_{i=1}^{\infty})=\pi((\sigma_i(g_1)\cdots\sigma_i(g_k)\phi_i(x_j))_{i=1}^{\infty})).$$

Now suppose $V\subseteq l^{\infty}(\mathbb{N},M_{d_i}(\mathbb{C}))$ ε-contains $\{((\rho_i(x_j)B)_{j=1}^{\infty}:\|B\|_{\infty}\leq 1\}$. Define a map $\Phi\colon l^{\infty}(\mathbb{N},M_{d_i}(\mathbb{C}))\to L^2(M_{d_i}(\mathbb{C}),\mathrm{tr})$ by

$$\Phi(f)=\sum_{g\in E,1\leq j\leq l}c_{gj}\sigma_i(g)f(j),$$

then our hypotheses imply that for all large i,

$$\Phi(V)\supseteq_{3\varepsilon,\|\cdot\|_2}\{qB:B\in\mathrm{Ball}(M_{d_i}(\mathbb{C}),\|\cdot\|_{\infty})\}.$$

Our methods to prove Theorem 3.4 can be used to complete the proof. $\square$

4. Definition of l^p-Dimension Using Vectors

In this section, we give a definition of the extended von Neumann dimension using vectors instead of almost equivariant operators. Thus may be conceptually simpler, as we do not have to deal with the technicalities involving changing domains inherit to the definition of $\mathrm{Hom}_{\Gamma}(\cdots)$. The definition is in much simpler and requires fewer preliminaries as well. However, for many theoretical purposes it will stil be easier to use the notion of almost equivariant operators. We will give this alternate defintion after the following Lemma.

Lemma 4.1. *Let V be a finite-dimensional Banach space spanned by vectors $v_1,\cdots,v_n$. Then for any $\varepsilon>0$, there is a $\delta>0$ so that if Y is a Banach space and $\xi_1,\cdots,\xi_n$ have the property that for all $c_1,\cdots,c_n\in\mathbb{C}$ with $\sum_{k=1}^n|c_k|\leq 1$,*

$$\left\|\sum_{k=1}^n c_k\xi_k\right\|\leq\delta+\left\|\sum_{k=1}^n c_kv_k\right\|,$$

then there is a $T\colon V\to Y$ with $\|T\|\leq 1$, such that

$$\|T(v_j)-\xi_j\|<\varepsilon.$$

Proof. Let $A\subseteq\{1,\cdots,n\}$ be such that $\{v_j:j\in A\}$ is a basis for V. Let $\widetilde{T}\colon V\to Y$ be defined by

$$\widetilde{T}(v_j)=\xi_j,$$

for $j\in A$. By finite-dimensionality, there is a $C(V)>0$ so that

$$\sum_{k=1}^n|c_j|\leq C(V)\left\|\sum_{j\in A}c_jv_j\right\|.$$

Thus our hypothesis implies that

$$\|\widetilde{T}\|\leq C(V)\delta+1.$$

Set $T = \frac{1}{1+C(V)\delta}\widetilde{T}$. For each $j \in \{1, \cdots, n\} \setminus A$ choose $c_k^{(j)}, k \in A$ so that

$$v_j = \sum_{k \in A} c_k^{(j)} v_j.$$

Then

$$\|T(v_j) - \xi_j\| = \left\| \frac{1}{C(V)\delta + 1} \sum_{k \in A} c_k^{(j)} \xi_k - \xi_j \right\| \leq$$

$$\sup_j |c_k^{(j)}| \left| 1 - \frac{1}{C(V)\delta + 1} \right| + \left\| \sum_{k \in A} c_k^{(j)} \xi_k - \xi_j \right\| \leq$$

$$\sup_j \left(|c_k^{(j)}| \left| 1 - \frac{1}{C(V)\delta + 1} \right| + \delta \right).$$

Clearly the right hand side can be made smaller than ε by choosing δ small. $\square$

Definition 4.2. Let X be a Banach space with a uniformly bounded action of a countable discrete group Γ and $\sigma_i \colon \Gamma \to \mathrm{Isom}(V_i)$ with V_i finite-dimensional. We let $\mathrm{Vect}_\Gamma(S, F, m, \delta, \sigma_i)$ be all vectors $(\xi_j)_{j=1}^m$ such that for all $c_{g_1, \cdots, g_m, j}$ with $\sum_{\substack{g_1, \cdots, g_m \in F \\ 1 \leq j \leq m}} |c_{g_1, \cdots, g_m, j}| \leq 1$, we have

$$\left\| \sum_{\substack{g_1, \cdots, g_m \in F \\ 1 \leq j \leq m}} c_{g_1, \cdots, g_m, j} \sigma_i(g_1) \cdots \sigma_i(g_m) \xi_j \right\| \leq \delta + \left\| \sum_{\substack{g_1, \cdots, g_m \in F \\ 1 \leq j \leq m}} c_{g_1, \cdots, g_m, j} g_1 \cdots g_m x_j \right\|.$$

Set

$$\mathrm{vdim}_\Sigma(S, F, m, \delta, \varepsilon, \rho) = \limsup_{i \to \infty} \frac{1}{\dim V_i} d_\varepsilon(\mathrm{Vect}_\Gamma(S, F, m, \delta, \sigma_i), \rho_{V_i}),$$

$$\mathrm{vdim}_\Sigma(S, \varepsilon, \rho) = \inf_{F, m, \delta} \mathrm{vdim}_\Sigma(S, F, m, \delta, \varepsilon, \rho),$$

$$\mathrm{vdim}_\Sigma(S, \rho) = \sup_{\varepsilon > 0} \mathrm{vdim}_\Sigma(S, \varepsilon, \rho).$$

Proposition 4.3. *Let X be a Banach space with a uniformly bounded action of a countable discrete group Γ and $\sigma_i \colon \Gamma \to \mathrm{Isom}(V_i)$ with V_i finite-dimensional. Then for any dynamically generating sequence S, and any product norm ρ,*

$$\dim_\Sigma(X, \Gamma) = \mathrm{vdim}_\Sigma(S, \rho).$$

Proof. Fix $e \in F \subseteq \Gamma$ finite, $m \in \mathbb{N}$, $\delta > 0$. Let $\delta' > 0$ to be determined. Suppose that $T \in \mathrm{Hom}_\Gamma(S, F, m, \delta, \sigma_i)$ and set $\xi_j = T(x_j)$. Then for all $c_{g_1, \cdots, g_m, j}$ with

$$\sum_{\substack{g_1, \cdots, g_m \in F \\ 1 \leq j \leq m}} |c_{g_1, \cdots, g_m, j}| \leq 1,$$

we have

$$\left\| \sum_{\substack{g_1,\cdots,g_m\in F\\ 1\leq j\leq m}} c_{g_1,\cdots,g_m,j}\sigma_i(g_1)\cdots\sigma_i(g_m)\xi_j \right\| \leq \delta + \left\| T\left(\sum_{\substack{g_1,\cdots,g_m\in F\\ 1\leq j\leq m}} c_{g_1,\cdots,g_m,j} g_1\cdots g_m\xi_j \right) \right\|$$

$$\leq \delta + \left\| \sum_{\substack{g_1,\cdots,g_m\in F\\ 1\leq j\leq m}} c_{g_1,\cdots,g_m,j} g_1\cdots g_m\xi_j \right\|.$$

So $(\xi_j)_{j=1}^m \in \mathrm{Vect}_\Gamma(S,F,m,\delta,\sigma_i)$ and $\mathrm{vdim} \leq \dim$.

For the opposite inequality, fix $e \in F \subseteq \Gamma$ finite $m \in \mathbb{N}, \delta' > 0$. By the lemma find $\delta > 0$ for $V = X_{F,m}, v_{g_1,\cdots,g_m,j} = g_1\cdots g_m\xi_j$, and $\varepsilon = \delta'$. Thus if $(\xi_j)_{j=1}^m \in \mathrm{Vect}_\Gamma(S,F,m,\delta,\sigma_i)$, then we can find a $T\colon X_{F,m} \to V_i$ with $\|T\| \leq 1$ and $\|T(g_1\cdots g_m x_j) - \sigma_i(g_1)\cdots\sigma_i(g_m)\xi_j\| < \delta'$. Thus

$$\|T(g_1\cdots g_m x_j) - \sigma_i(g_1)\cdots\sigma_i(g_m)T(x_j)\| < 2\delta',$$

and this proves the opposite inequality.

□

5. The Case of Finite-Dimensional Representations Revisited

In the section we prove that if Γ is an infinite sofic group, and Σ is a sofic approximation of Γ, then for any finite-dimensional representation V of Γ we have

$$\dim_{\Sigma,l^p}(V,\Gamma) = 0.$$

The method is based on passing to an action of the group on a measure space, and then using that the corresponding equivalence relations contains an action of $\mathbb{Z}$.

We shall first work trivial action of Γ on $\mathbb{C}$. For this, fix a sofic group Γ and a sofic approximation Σ , for $S = \{1\}$, and the trivial action of Γ on $\mathbb{C}$, note that $T \to T(\{1\})$, identifies $\mathrm{Hom}_{\Gamma,p}(S,F,m,\delta,\sigma_i)$ with all vectors $\xi \in l^p(d_i)$ such that

$$\|\sigma_i(g)\xi - \xi\|_p < \delta$$

for all $g \in F$.

For the proof of the next Lemma, we will also need the concept of a sofic approximation of an equivalence relation. Let us recall some preliminary definitions.

Definition 5.1. A discrete equivalence relation, is a triple $(\mathcal{R}, X, \mu)$ where (X,μ) is a standard probability space, $\mathcal{R} \subseteq X \times X$ is a subset such that for all the relation $x \sim y$ if $(x,y) \in \mathcal{R}$ is an equivalence relation and such that for almost every $x \in X, \{y : (x,y) \in \mathcal{R}\}$ is countable. We say that $\mathcal{R}$ is *measure-preserving* if for all $A \subseteq \mathcal{R}$, such that $x \mapsto |\{y \in X : (x,y) \in A\}|$ measurable, we have that $y \mapsto |\{x \in X : (x,y) \in A\}|$ is measurable and

$$\int_X |\{y \in X : (x,y) \in A\}|\, d\mu(x) = \int_X |\{x \in X : (x,y) \in A\}|\, d\mu(y).$$

We shall denote the above measure by $\overline{\mu}(A)$, defined on the subsets $A \subseteq \mathcal{R}$ so that $x \mapsto |\{y \in X : (x,y) \in A\}|$ is measurable. We say that $\mathcal{R}$ is *ergodic* if for all $f\colon X \to \mathbb{C}$ measurable with $f(x) = f(y)$ for $\overline{\mu}$-almost every $(x,y) \in \mathcal{R}$, there is a $\lambda \in \mathbb{C}$ so that $f(x) = \lambda$ almost everywhere.

The main example of relevance for us is given by taking a countable discrete group Γ with a measure-preserving action on a standard probablity space (X,μ). In this case $\mathcal{R}_{\Gamma\curvearrowright(X,\mu)}=\{(x,gx):g\in\Gamma\}$. Such an action is *free* if for all $g\in\Gamma\setminus\{e\}$, $\mu(\{x\in X:gx=x\})=0$.

Definition 5.2. Let $(\mathcal{R},X,\mu)$ be a discrete, measure-preserving equivalence relation. A *partial morphism* is a bi-measurable bijection map $\phi\colon \operatorname{dom}(\phi)\to\operatorname{ran}(\phi)$ where $\operatorname{dom}(\phi),\operatorname{ran}(\phi)\subseteq X$ are measurable and $(x,\phi(x))\in\mathcal{R}$ for almost every $x\in\operatorname{dom}(\phi)$. We let ϕ^{-1} be the partial morphism with $\operatorname{dom}(\phi^{-1})=\operatorname{ran}(\phi),\operatorname{ran}(\phi^{-1})=\operatorname{dom}(\phi)$ and $\phi(\phi^{-1}(x))=x$ for $x\in\operatorname{ran}(\phi)$. If $\phi,\psi\in[[\mathcal{R}]]$, we let $\phi\circ\psi$ be the partial morphism with $\operatorname{dom}(\phi\circ\psi)=\{x\in\operatorname{dom}(\psi):\psi(x)\in\operatorname{dom}(\phi)\}$, and $(\phi\circ\psi)(x)=\phi(\psi(x))$. If $A\subseteq X$ is measurable, we let $\operatorname{Id}_A\colon A\to A$ be the partial morphism which is the identity on A. We let $[[\mathcal{R}]]$ be the set of all partial morphisms of $\mathcal{R}$, we let $[\mathcal{R}]$ be the set of all partial morphisms of $\mathcal{R}$ so that $\mu(\operatorname{dom}(\phi))=1$. For every $\phi\in[[\mathcal{R}]]$ define the operator $v_\phi\in B(L^2(\mathcal{R},\overline{\mu}))$ defined by $v_\phi f(x,y)=\chi_{\operatorname{ran}(\phi)}(x)f(\phi^{-1}x,y)$. Let $L(\mathcal{R})$ be the von Neumann subalgebra of $B(L^2(\mathcal{R},\overline{\mu}))$ generated by $\{v_\phi:\phi\in[[\mathcal{R}]]\}$. If $\mathcal{R}$ is given by a free measure-preserving action of Γ, and $\alpha_g\colon X\to X$ for $g\in\Gamma$ is defined by $\alpha_g(x)=gx$, we let $u_g=v_{\alpha_g}$.

We let $\tau\colon L(\mathcal{R})\to\mathbb{C}$ be defined by $\tau(x)=\langle x\chi_\Delta,\chi_\Delta\rangle$ where $\Delta=\{(x,x):x\in X\}$.

For the next definition, we need to note the following example. For $n\in\mathbb{N}$, let R_n be the equivalence relation on $\{1,\cdots,n\}$ declaring all points to be equivalent. By the process described we thus embed $[[R_n]]$ into $B(l^2(n))\cong M_n(\mathbb{C})$.

Definition 5.3. Let Γ be a countable discrete sofic group with sofic approximation $\Sigma=(\sigma_i\colon\Gamma\to S_{d_i})$. Let Γ have a free, measure-preserving action on a standard probability space (X,μ). A *sofic approximation of $\mathcal{R}$ extending* Σ is a sequence of maps $\Sigma'=(\rho_i\colon L(\mathcal{R})\to M_{d_i}(\mathbb{C}))$ such that

$$\rho_i(u_g)=\sigma_i(g),\ \text{for all } g\in\Gamma$$
$$\rho_i(\phi)\in[[R_n]]$$
$$\text{for all } A\subseteq X \text{ measurable, there exists } A_i\subseteq\{1,\cdots,d_i\} \text{ so that } \rho_i(\operatorname{Id}_A)=\operatorname{Id}_{A_i}$$
$$\operatorname{tr}\circ\rho_i(x)\to\tau(x) \text{ for all } x\in L(\mathcal{R})$$
$$\sup_i\|\rho_i(x)\|_\infty<\infty \text{ for all } x\in L(\mathcal{R})$$
$$\|P(\rho_i(x_1),\cdots,\rho_i(x_n))-\rho_i(P(x_1,\cdots,x_n))\|_2\to 0,$$

for all $x_1,\cdots,x_n\in L(\mathcal{R})$ and all $*$-polynomials in n noncommuting variables.

Lemma 5.4. *Let Γ be a countable discrete sofic group with a sofic approximation Σ. Let $\Gamma\curvearrowright(X,\mu)$ be a free, ergodic, measure-preserving action on a standard probablity space (X,μ) such that there is a sofic approximation (still denoted Σ) of $\mathcal{R}_{\Gamma\curvearrowright(X,\mu)}$ extending the sofic approximation of Γ. Fix $\phi\in[[\mathcal{R}]]$, and $\eta>0$. Then there is a $F\subseteq\Gamma$ finite, $m\in\mathbb{N}$, $\delta>0$ and $C_i\subseteq\{1,\cdots,d_i\}$ with $|C_i|\geq(1-\eta)d_i$ so that for the trivial representation of Γ on $\mathbb{C}$, and $T\in\operatorname{Hom}_{\Sigma,p}(\{1\},F,m,\delta,\sigma_i)$ with $\xi=T(1)$ we have*

$$\|\sigma_i(\phi)\xi-\sigma_i(\operatorname{Id}_{\operatorname{ran}(\phi)})\xi\|_{l^p(C_i)}<\eta,$$

for all large i.

Proof. Let $\{A_g : g \in \Gamma\}$ be a partition of $\mathrm{ran}(\phi)$ so that

$$v_\phi = \sum_{g\in\Gamma} \mathrm{Id}_{A_g} u_g.$$

Choose $F \subseteq \Gamma$ finite so that

$$\left\| v_\phi - \sum_{g\in F} \mathrm{Id}_{A_g} u_g \right\|_2 < \eta.$$

We find $|C_i|$ with $|C_i| \geq (1-2\eta)d_i$ so that

$$\chi_{C_i}\sigma_i(\mathrm{Id}_{A_g})\chi_{C_i}\sigma_i(\mathrm{Id}_{A_h})\chi_{C_i} = 0, \text{for } g \neq h \text{ in } F$$

$$\sigma_i(\phi) = \sum_{g\in F} \sigma_i(\mathrm{Id}_{A_g})\sigma_i(g) \text{ on } C_i.$$

Thus on C_i,

$$\sigma_i(\phi)\xi = \sum_{g\in F} \sigma_i(\mathrm{Id}_{A_g})\sigma_i(g)\xi,$$

$$\sigma_i(\mathrm{Id}_{\mathrm{ran}(\phi)})\xi = \sum_{g\in F} \sigma_i(\mathrm{Id}_{A_g})\xi,$$

so

$$\|\sigma_i(\phi)\xi - \sigma_i(\mathrm{Id}_{\mathrm{ran}(\phi)})\xi\|_{l^p(C_i)} \leq |F|\delta,$$

so if $\delta < \frac{\eta}{|F|}$, our claim is proved.

□

Lemma 5.5. *Let Γ be a countably infinite discrete sofic group with sofic approximation Σ. Then for the trivial representation of Γ on $\mathbb{C}$, we have*

$$\dim_{\Sigma,l^p}(\mathbb{C},\Gamma) = 0.$$

Proof. Let $\mathcal{R}$ be the equivalence relation induced by the Bernoulli action of Γ on $(X,\mu) = (\{0,1\},u)^\Gamma$, u being the uniform measure. Extend Σ to a sofic approximation of $[[\mathcal{R}]]$, since Γ is an infinite group, we know that for all $n \in \mathbb{N}$, there is a subequivalence relation $\mathcal{R}_n$, generated by a free probability measure preserving action of $\mathbb{Z}/n\mathbb{Z}$ on (X,μ). Let $\alpha \in [\mathcal{R}_n]$ generate the action of $\mathbb{Z}/n\mathbb{Z}$ on (X,μ). Fix $\eta > 0$, choose a finite subset $F \subseteq \Gamma, \delta > 0$ and subsets $C_i \subseteq \{1,\cdots,d_i\}$ with $|C_i| \geq (1-d_i)\eta$ so that if $T \in \mathrm{Hom}_\Gamma(\{1\},F,1,\delta,\sigma_i)$ and $\xi = T(1)$, then

$$\|\sigma_i(\alpha)^j\xi - \xi\|_{l^p(C_i)} < \eta, \text{for } 1 \leq j \leq n-1$$

for all large i. We may assume that there are $A_i \subseteq \{1,\cdots,d_i\}$ with $\frac{|A_i|}{d_i} \to \frac{1}{n}$, so that

$$\{\sigma_i(\alpha)^j(A_i) : 0 \leq j \leq n-1\}, \text{ are a disjoint family.}$$

$$\sigma_i(\alpha)\big|_{\{1,\cdots,d_i\}\setminus\bigcup_{j=0}^{n-1}\sigma_i(\alpha)^j(A_i)} = \mathrm{Id}.$$

Let

$$\eta = \sum_{i=1}^n \sigma_i(\alpha)^j\chi_{A_i}\xi = \sum_{i=1}^n \chi_{\sigma_i(\alpha)^j(A_i)}\sigma_i(\alpha)^j\xi.$$

Set $D_i = C_i \cap \bigcup_{j=0}^{n-1}\sigma_i(\alpha)^j(A_i)$, then

$$\chi_{D_i}\eta - \chi_{D_i}\xi = \sum_{i=1}^n \chi_{D_i\cap\sigma_i(\alpha)^j(A_i)}(\sigma_i(\alpha)^j\xi - \xi),$$

so

$$\|\chi_{D_i}\eta - \chi_{D_i}\xi\|_p \le \eta n.$$

Thus

$$\dim_\Sigma(\{1\}, \eta n, \Gamma) \le \frac{1}{n} + 2\eta n.$$

Letting $\eta \to 0$, and then $n \to \infty$ completes the proof.

□

Theorem 5.6. *Let Γ be a countably infinite sofic group with sofic approximation Σ. The for any represenation of Γ on a finite-dimensional vector space V, and for all $1 \le p < \infty$,*

$$\dim_{\Sigma,l^p}(V, \Gamma) = 0$$

Proof. By [9], we may assume that

$$\{|\Lambda| : \Lambda \subseteq \Gamma \text{ is finite}\},$$

is bounded, and that every element of Γ has finite order. Also, the usual tricks imply that we may assume that V is a Hilbert space and Γ acts by unitaries. Let M be greater than $|\Lambda|$ for any finite subgroup of Γ. Choose $\varepsilon > 0$ so that if U is a unitary on a Hilbert space and

$$\|U - 1\| < \varepsilon,$$

then $U^M \ne 1$ unless $U = 1$. Since $\overline{\pi(\Gamma)}$ is compact, we may find an infinite sequence g_n of distinct elements of Γ so that

$$\|\pi(g_n) - 1\| < \varepsilon.$$

If

$$\Lambda = \langle g_n : n \in \mathbb{N}\rangle,$$

our assumptions then imply that Λ is an infinite subgroup of Γ which acts trivially. Thus by the preceding Lemma,

$$\dim_{\Sigma,l^p}(V, \Gamma) \le \dim_{\Sigma,l^p}(V, \Lambda) = 0.$$

□

6. Closing Remarks

The results in this paper answer most of the conjectures asked in [9]. In particular, we answer all the conjectures affirmatively except for the following.

Conjecture 1. *Let Γ be a an amenable group, and $Y \subseteq l^p(\Gamma)^{\oplus n}$, for some $n \in \mathbb{N}$. Let $\dim^G_{l^p}(Y, \Gamma)$ be l^p-Dimension as defined by Gornay. Then for any sofic approximation Σ of Γ we have*

$$\dim^G_{l^p}(Y, \Gamma) = \dim_{\Sigma,l^p}(Y, \Gamma).$$

Conjecture 2. *Let $2 < p < \infty$, and let Γ be a countable discrete sofic group with sofic approximation Σ. Then for all $n \in \mathbb{N}$,*

$$\dim_{\Sigma,l^p}(l^p(\Gamma)^{\oplus n}, \Gamma) = \underline{\dim}_{\Sigma,l^p}(l^p(\Gamma)^{\oplus n}, \Gamma)$$

Little progress has been made on Conjectures 1,2. It may be quite possible that our definition is simply not the right way to look at von Neumann dimension for the action of Γ on $l^p(\Gamma)^{\oplus n}$ if $2 < p < \infty$.

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